Flowing to four dimensions

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ABSTRACT: We analyze the properties of a model with four-dimensional brane-localized Higgs type potential of a six dimensional scalar field satisfying the Dirichlet boundary condition on the boundary of a transverse two-dimensional compact space. The regularization of the localized couplings generates classical renormalization group running. A tachyonic mass parameter grows in the infrared, in analogy with the QCD gauge coupling in four dimensions. We find a phase transition at a critical value of the bare mass parameter such that the running mass parameter becomes large in the infrared precisely at the compactification scale. Below the critical coupling, the theory is in symmetric phase, whereas above it spontaneous symmetry breaking occurs. Close to the phase transition point there is a very light mode in the spectrum. The massive Kaluza-Klein spectrum at the critical coupling becomes independent of the UV cutoff.

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JHEP03(2006)085

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1. Introduction

Dynamical generation of small mass scales via logarithmic renormalization group running and dimensional transmutation is a common feature of many field theories. This phenomenon often occurs in theories which, in the limit of vanishing couplings, possess massless degrees of freedom. Once small couplings are turned on in UV, a theory may become strongly coupled in IR, with the corresponding IR scale serving as the mass scale of the low energy effective theory. A notable example of such a case is of course QCD.

It is less common that massless or very light degrees of freedom emerge upon turning on small couplings in a theory which originally had heavy states only. Unless the existence of massless states is dictated by a symmetry (e.g., Goldstone modes), this requires some sort of IR–UV mixing. In this paper we present simple examples of this sort. In general terms, our models have two widely separated high energy scales, the UV cutoff scale Λ and the "intermediate scale" R^{-1} (the reason for the latter notation will become clear shortly). Once the small coupling $\mu = \mu(\Lambda)$ is turned on at the UV cutoff scale, it experiences

renormalization group running and increases towards low energies. There exists a (still small) critical value $\mu(\Lambda) = \mu_c$ such that the running coupling blows up precisely at the scale R^{-1}, i.e., $\mu(R^{-1}) = \infty$. At this point a massless mode appears in the spectrum.

For μ slightly smaller than μ_c, the light mode has positive mass squared proportional to $(\mu_c^2 - \mu^2)$, whereas at μ slightly above μ_c the low energy theory is in a symmetry breaking phase. Thus, $\mu(\Lambda) = \mu_c$ is the point of the second order phase transition.

These features are reminiscent of phenomena appearing in string theory [1], where strong coupling is associated with the presence of new light degrees of freedom in the spectrum at particular points in the moduli space. Our models are much simpler, and, in fact, the features described above occur at the level of classical field theory. Specifically, we consider models which contain six-dimensional scalar fields whose couplings are localized on a four-dimensional brane. Their peculiar property, due to the codimension-two nature of the couplings, is that the would-be mass parameter μ^2 is actually dimensionless and experiences a classical (from the field theory viewpoint) running[1], coming from the regularization of UV logarithmic divergences [2, 3] (see also ref. [4]). This regularization has a natural interpretation as finite transverse size $\epsilon = \Lambda^{-1}$ of the brane. Remarkably enough, the coupling becomes stronger in the infrared provided that the coupling is negative or, equivalently, the 4d localized mass is tachyonic. We consider the models on a disk in transverse dimensions, with the Dirichlet boundary condition. The radius R of the disk serves as the intermediate scale. As described above, we find that the critical value of the bare coupling, $\mu(\Lambda) = \mu_c$ (such that $\mu(R^{-1}) = \infty$) is the point of the second order phase transition, at which there is a massless mode.

From the viewpoint of phenomenology with extra dimensions, our models generate effective four-dimensional theories with small mass scales out of six-dimensional theories whose energy scales (both the UV scale $\Lambda \equiv \epsilon^{-1}$ and intermediate scale R^{-1}) may be very high — say, of the order of the Planck mass. This separation of scales is somewhat similar to what happens in warped models [7, 8], but, unlike in the latter, the Kaluza–Klein excitations in our models have very large masses. In our models as they stand, the (bare) couplings are free parameters, so the choice $\mu \approx \mu_c$ (and hence the theory near the phase transition point) requires fine tuning[2]. It would be interesting to see if there exists a dynamical mechanism driving the theory to the critical coupling (more generally, driving the parameters to the point where $\mu(R^{-1}) = \infty$).

In section 2 we present the simplest model realizing this mechanism. Section 3 contains a detailed six dimensional description of the spectrum and interactions near the critical coupling. In section 4 we show that the light mode present near the critical coupling has standard four-dimensional mass and interactions at low energies. In section 5 we discuss another property of the model, namely, that the massive Kaluza-Klein spectrum becomes independent of the UV cutoff precisely at the critical coupling. Section 6 presents a generalization of the simple model discussed previously, by adding a brane scalar which mixes with the bulk scalar. We show that in this case a new critical coupling arises, above

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[1] The classical running was recently discussed in connection with neutrino masses in ref. [5]. The phenomenon is well-known in two-dimensional quantum mechanics, see e.g. ref. [6].

[2] Somewhat analogous fine tuning in theories with codimension two has been discussed in ref. [9].

which the scalar potential appears destabilized. Section 7 shows that the new critical coupling, determined by the renormalization group arguments in section 6, can again be understood in terms of a phase transition (in a regularized brane setup and with appropriate boundary conditions). We conclude with brief remarks and prospects for further studies.

2. The model

The theory under consideration is a scalar theory in 6d space with two flat compact codimensions, with no scalar potential in the bulk and Higgs-type scalar potential on a brane situated at the origin of the compact space, described by the action[3]

$$
S = \int d^4x d^2y \left[\frac{1}{2} (\partial_M \phi)^2 - V_\delta(\phi) \right] ,
$$

$$
V_\delta(\phi) = \left(-\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right) \cdot \delta^2(y) . \tag{2.1}
$$

We resolve the singularity at $y = 0$ by introducing a disk $r < \epsilon$ supporting the potential,

$$
V(\phi) = \frac{1}{\pi \epsilon^2} \left(-\frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right) \quad \text{for } 0 < r < \epsilon , \tag{2.2}
$$

$$
V(\phi) = 0 \quad \text{for } \epsilon < r < R .
$$

The parameter μ^2 is dimensionless and may be considered a coupling constant. We will assume $\mu^2 \ll 1$. Goldberger and Wise [2] argued that the coupling runs with the energy scale Q, i.e.,

$$
\mu^2(Q) = \frac{\mu^2}{1 + \frac{\mu^2}{2\pi} \ln \frac{Q}{\Lambda}} , \tag{2.3}
$$

where Λ is the UV cutoff and $\mu^2 \equiv \mu^2(\Lambda)$ is the coupling constant entering the potential (2.2); we identify $\Lambda = \epsilon^{-1}$. Equation (2.3) implies that the coupling $\mu^2(Q)$ grows in the infrared.

We consider a theory on a disk of radius R, which is assumed to be small (but $R \gg \epsilon$). Let us impose the Dirichlet boundary condition

$$
\phi(r = R) = 0 .
$$

If the coupling vanishes, or, more generally, if the running coupling at the compactification scale is small, $\mu^2(R^{-1}) \ll 1$, the theory is fully 6-dimensional: there are no zero or light modes, whereas the Kaluza–Klein states have masses of order R^{-1}.

The question is what happens when $\mu^2(R^{-1})$ hits the infrared pole. In other words, what does this theory describe when the bare coupling μ^2 is equal or close to its critical value

$$
\mu_c^2 = \frac{2\pi}{\ln \frac{R}{\epsilon}} . \tag{2.4}
$$

---

[3]We use the convention $(+, -, -, -, -, -)$ for the metric.

This is a question of the classical field theory, as the running (2.3) occurs at the classical field theory level. Our claim is that very close to the critical coupling there is a light mode, whose mass is proportional to $\mu_c^2 - \mu^2$ and vanishes just at the critical coupling. At $\mu^2 > \mu_c^2$ this mode is tachyonic, and the field develops vacuum expectation value. Furthermore, the low energy theory may be described fully in four-dimensional terms, and it is a four-dimensional scalar theory with (without) the Higgs mechanism for $\mu^2 > \mu_c^2$ ($\mu^2 < \mu_c^2$). This result can be understood qualitatively by noticing that the Dirichlet boundary condition forces all modes to acquire a mass, in the spirit of the Scherk–Schwarz mechanism [10]. Then from a naive 4d viewpoint, the mass of a would-be zero mode has both the contribution due to the boundary condition, and an additional contribution, of opposite sign, coming from the localized tachyonic term,

$$m_{(0),\text{naive}}^2 = \frac{z_0^2}{R^2} - \frac{\mu^2}{\pi R^2} \ , \tag{2.5}$$

where z_0 is the first zero of the Bessel function J_0 and the factor $1/(\pi R^2)$ in the second term comes from the KK expansion of the zero mode. Equation (2.5) indeed predicts phase transition, but for $\mu_c^2 = (\pi z_0^2)$. The correct value, eq. (2.4), however, contains an inverse logarithmic factor coming from the running and the correct expression for the mass is actually slightly more involved than the simple guess (2.5). As follows from (2.4) and explained in more detail in the following section, μ_c^2 can (and actually, for perturbative treatment, has to) be small, which requires that $\ln(R/\epsilon)$ is large. Thus, it is legitimate to make use of the leading-log approximation and this is what we are going to do. Incidentally, in the theory on a disk, the formula for the running (2.3) is also valid in the leading-log approximation only.

3. Six-dimensional description of the phase transition

3.1 At the critical coupling

Let us see that at the critical coupling, there exists a massless mode about the background $\phi = 0$. In this background, the massless mode obeys the free massless equation away from the brane

$$\Delta^{(2)}\phi = 0 \ , \quad r > \epsilon$$

and a free equation with negative mass squared inside the resolved brane,

$$\Delta^{(2)}\phi + \frac{\mu^2}{\pi\epsilon^2}\phi = 0 \ , \quad r < \epsilon \ .$$

The massless mode is in the s-wave, so the solution inside the resolved brane is

$$\phi(r) = f_0 \left(1 - \frac{\mu^2}{4\pi\epsilon^2} \cdot r^2 + \cdots\right) \ , \qquad r < \epsilon \ , \tag{3.1}$$

where f_0 is a small amplitude and dots stand for terms of higher order in μr. Note that the second term here is suppressed with respect to the first term at all $r < \epsilon$, since μ^2 is small. Higher order terms are even more suppressed.

The solution outside the brane is

$$\phi(r) = a \ln \frac{R}{r} \ , \quad r > \epsilon \ , \tag{3.2}$$

where we used the Dirichlet boundary condition at $r = R$. Matching conditions for ϕ and $d\phi/dr$ at $r = \epsilon$ give

$$a \ln \frac{R}{\epsilon} = f_0 \ ,$$
$$a = \frac{\mu^2}{2\pi} f_0 \ .$$

These are consistent right at the critical coupling, $\mu^2 = \mu_c^2$. We conclude that at this value of the coupling, there indeed exists a zero mass state.

3.2 The mass of the light state near the critical coupling

To calculate the mass of the light state at μ^2 close to (and slightly below) μ_c^2, still in the background $\phi = 0$, one has to solve the following equation away from the brane

$$\Delta^{(2)}\phi + p^2\phi = 0 \ , \quad r > \epsilon \ , \tag{3.3}$$

where p^2 is the 4d mass. Inside the brane, the term with 4-momentum in the field equation is tiny, so the solution still has the form (3.1). For $p^2 \ll R^{-2}$, the s-wave solution to eq. (3.3) is

$$\phi(r) = a \left[\ln \frac{R}{r} - \frac{p^2 r^2}{4} \ln \frac{R}{r} + \frac{p^2}{4}(R^2 - r^2)\right] \ . \tag{3.4}$$

The terms suppressed by higher powers of $p^2 R^2$ and/or $p^2 r^2$ are irrelevant.

Now the matching conditions for ϕ and $d\phi/dr$ at $r = \epsilon$ are

$$f_0 = a \left(\ln \frac{R}{\epsilon} + \frac{p^2 R^2}{4}\right) \ ,$$
$$\frac{\mu^2}{2\pi} f_0 = a \ . \tag{3.5}$$

This gives for the four-dimensional mass squared

$$p^2 \equiv m_{(4)}^2 = \frac{4}{R^2}\left(\frac{2\pi}{\mu^2} - \ln \frac{R}{\epsilon}\right) = \frac{8\pi}{R^2} \frac{1}{\mu^2(R^{-1})} \ , \tag{3.6}$$

where $\mu^2(R^{-1})$ is the running coupling evaluated at the compactification scale R^{-1}. Equivalently, we can write

$$m_{(4)}^2 = \frac{8\pi}{R^2}\left(\frac{1}{\mu^2} - \frac{1}{\mu_c^2}\right) = \frac{8\pi}{R^2} \frac{\mu_c^2 - \mu^2}{\mu_c^4} \ . \tag{3.7}$$

(hereafter we consider the linear order in $(\mu_c^2 - \mu^2)$). Notice that the expression (3.6) for the 4d mass contains only the renormalized coupling $\mu^2(R^{-1})$, and not the bare coupling μ^2 itself. At the critical coupling, the light mode possesses scale invariance properties[4].

3.3 The Higgs expectation value

At $\mu^2 > \mu_c^2$ the mass (3.7) becomes tachyonic, and the scalar field develops a vacuum expectation value. To obtain it, we have to solve the field equation inside the brane,

$$\Delta^{(2)}\phi - \frac{\partial V}{\partial \phi} = 0 , \quad r < \epsilon \tag{3.8}$$

and match the solution to the outside profile given by eq. (3.2). The field ϕ changes inside the brane only slightly, so in eq. (3.8) one can use a constant value of $V' \equiv \frac{\partial V}{\partial \phi}$, evaluated at $\phi(r = 0)$. Proceeding as in subsection 3.1, one obtains the inside solution

$$\phi = \phi_0 + \frac{1}{4}V'(\phi_0) \cdot r^2 . \tag{3.9}$$

The second term here is indeed small compared to the first one for the same reason as in subsection 3.1.

Matching this solution to (3.2), one finds the equation[5] for ϕ_0,

$$\phi_0 = -\frac{\epsilon^2}{2}V'(\phi_0) \cdot \ln \frac{R}{\epsilon} .$$

This gives

$$1 = \frac{1}{2\pi} \ln \frac{R}{\epsilon} \cdot (\mu^2 - \lambda \phi_0^2) .$$

Recalling the definition of μ_c^2 one gets finally

$$\phi_0^2 = \frac{\mu^2 - \mu_c^2}{\lambda} . \tag{3.10}$$

Note that the solution outside the brane is simply

$$\phi_c(r) = \phi_0 \cdot \frac{\ln \frac{R}{r}}{\ln \frac{R}{\epsilon}} \tag{3.11}$$

and that inside the brane the solution is approximately constant and equal to $\phi_c = \phi_0$.

Let us calculate the mass of the light state (the "Higgs boson") in the background ϕ_c. Writing $\phi(r, p) = \phi_c(r) + \xi(r, p)$, one has in the inner region, neglecting the term with four-momentum and recalling that ϕ_c equals to ϕ_0 up to small corrections,

$$\Delta^{(2)}\xi - V''(\phi_0)\xi = 0 , \quad r < \epsilon .$$

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[4]For example, by using the quadratic part of the effective action and the solution of the field equations (3.1) – (3.2) , it can be shown that precisely at the critical coupling $\int_0^R dr \, r \, T_M^M(r) = 0$, where $T_{MN} = \partial_M \phi \partial_N \phi - \eta_{MN}[\frac{1}{2}(\partial \phi)^2 - V(\phi)]$ is the energy momentum tensor. Classically, the quadratic part of the action is naively scale invariant for any coupling μ. Due to the regularization of the delta function, this scale invariance is really valid only at the critical coupling, in analogy with quantum field theories at the fixed points of the beta functions and anomalous dimensions.

[5]The latter equation is in fact valid for arbitrary (but still small) μ^2 exceeding μ_c^2, and not only near the critical point. Thus, the result (3.10) is also valid for any μ^2 above μ_c^2.

In the outer region, ξ obeys free scalar equation, so the solution for $p^2 \ll R^{-2}$ still has the form (3.4). Proceeding as in subsection 3.2, one obtains for the Higgs mass

$$p^2 \equiv m_\xi^2 = \frac{8\pi}{R^2} \frac{\mu_c^2 + \pi\epsilon^2 V''(\phi_0)}{\mu_c^4} \ .$$

With ϕ_0 given by eq. (3.10) one obtains finally

$$m_\xi^2 = \frac{16\pi}{R^2} \cdot \frac{\mu^2 - \mu_c^2}{\mu_c^4} \ . \tag{3.12}$$

This expression is valid for small $(\mu^2 - \mu_c^2)$.

Let us now extend the model by considering the scalar field in some representation of a global symmetry group G. Then the vev $\phi_c(r)$ breaks this symmetry, and 4d Goldstone bosons appear. The Goldstone excitations have the form $\phi(x^\mu, r) = \exp(i\pi^a(x)T^a)\phi_c(r)$ where T^a are broken generators of G and $\pi^a(x^\mu)$ are massless 4d Goldstone fields. The low energy effective 4d action for the Goldstone fields takes the form of the standard sigma-model action,

$$S_{eff} = \sigma^2 \int d^4x \, \frac{1}{4} \left[(\partial_\mu \pi^a)^2 + \pi \cdot \partial_\mu \pi \cdot \partial^\mu \pi + \cdots \right] \ ,$$

where the effective 4d Higgs vev is

$$\sigma^2 = \int d^2y \, \phi_c^2 \ .$$

The contribution to this integral mainly comes from the outside region $\epsilon < r < R$. By using eq. (3.11), one obtains

$$\sigma^2 = 2\pi \int_0^R r dr \left(\phi_0 \cdot \frac{\ln \frac{R}{r}}{\ln \frac{R}{\epsilon}} \right)^2$$

$$= \frac{\pi}{2} \frac{\phi_0^2 R^2}{\ln^2(\frac{R}{\epsilon})} = \frac{R^2}{8\pi\lambda} \cdot \mu_c^4 \, (\mu^2 - \mu_c^2) \ . \tag{3.13}$$

If there are gauge fields in the bulk (with Neumann boundary condition for A_μ at $r = R$), then ϕ plays the role of the Higgs field. The light modes of the gauge fields are constant in extra dimensions, so that they obtain the masses $M_V^2 = g_{(4)}^2 \sigma^2$, where $g_{(4)}$ is the 4d gauge coupling and the effective 4d Higgs vev is given precisely by eq. (3.13). Notice that $M_V^2/m_\xi^2 \gg 1$ if $\lambda \sim 1/\Lambda^4$, whereas the more interesting result $M_V^2/m_\xi^2 \leq 1$ is valid if $\lambda \geq R^4 \mu_c^8$.

4. Four-dimensional effective theory

Let us see how the results of section 3 compare to the standard 4d effective low energy theory. It is clear from eqs. (3.2), (3.4), (3.11) that up to small corrections, the interesting field configuration away from the brane is

$$\phi(x^\mu, r) = \sigma(x^\mu) \cdot \zeta(r) \ , \tag{4.1}$$

where

$$\zeta(r) = \sqrt{\frac{2}{\pi R^2}} \ln \frac{R}{r} .$$

Resolving the brane is equivalent to defining

$$\zeta(0) = \zeta(\epsilon) = \sqrt{\frac{2}{\pi R^2}} \ln \frac{R}{\epsilon}$$
$$= \frac{2\sqrt{2\pi}}{\mu_c^2 R} . \tag{4.2}$$

We are interested in the effective theory of the 4d field $\sigma(x^\mu)$. The wave function ζ is normalized to unity,

$$\int d^2y \; \zeta^2 = 1$$

and therefore $\sigma(x)$ has a canonical kinetic term. Its effective 4d potential is obtained by plugging (4.1) and (4.2) into the potential (2.1) and the transverse kinetic energy term. One finds

$$V_{eff}(\sigma) = \frac{m_{(4)}^2}{2} \sigma^2 + \frac{\lambda_{(4)}}{4} \sigma^4 ,$$

where

$$\lambda_{(4)} = \lambda[\zeta(0)]^4 = \frac{64\pi^2}{\mu_c^8} \frac{\lambda}{R^4}$$

and

$$m_{(4)}^2 = -\mu^2[\zeta(0)]^2 + 2\pi \int_\epsilon^R rdr \; (\zeta')^2 .$$

The value of the latter is precisely the same as in (3.7), which establishes the correspondence between the 6d and the 4d approaches in the unbroken phase. For $m_{(4)}^2 < 0$, the 4d expressions for the vev and the Higgs mass are $\sigma^2 = -\frac{m_{(4)}^2}{\lambda_{(4)}}$ and $m_\xi^2 = -2m_{(4)}^2$. These coincide with (3.13) and (3.12), respectively, so that the correspondence exists in the broken phase as well.

5. Massive spectrum near the critical coupling

Let us now work out the massive spectrum at and slightly below the critical coupling. In this case the background is $\phi = 0$ and by defining the wave functions $\phi(r,\theta) = e^{il\theta} \chi_l(r)$, we get the Schrödinger equations

$$\chi_l'' + \frac{1}{r} \chi_l' + \left(p^2 + \frac{\mu^2}{\pi\epsilon^2} - \frac{l^2}{r^2} \right) \chi_l = 0 \quad , \quad \text{for } r < \epsilon ,$$
$$\chi_l'' + \frac{1}{r} \chi_l' + \left(p^2 - \frac{l^2}{r^2} \right) \chi_l = 0 \quad , \qquad \text{for } r > \epsilon . \tag{5.1}$$

The solutions for the positive mass squared (positive p^2) wave functions, satisfying the Dirichlet boundary condition at $r = R$ are

$$\chi_l(r) = A \, J_l\left(\sqrt{p^2 + \frac{\mu^2}{\pi \epsilon^2}} r\right) \quad , \qquad \text{for } r < \epsilon \, ,$$

$$\chi_l(r) = B \left[J_l(pr) - \frac{J_l(pR)}{N_l(pR)} N_l(pr) \right] \quad , \quad \text{for } r > \epsilon \, , \tag{5.2}$$

where J_l and N_l are the Bessel functions of the first and second kind. The matching condition of the logarithmic derivative of the wave function, which defines the massive spectrum, is

$$\sqrt{p^2 + \frac{\mu^2}{\pi \epsilon^2}} \; \frac{J_l'\left(\sqrt{p^2 \epsilon^2 + \frac{\mu^2}{\pi}}\right)}{J_l\left(\sqrt{p^2 \epsilon^2 + \frac{\mu^2}{\pi}}\right)} \; = \; p \; \frac{J_l'(p\epsilon) - \frac{J_l(pR)}{N_l(pR)} N_l'(p\epsilon)}{J_l(p\epsilon) - \frac{J_l(pR)}{N_l(pR)} N_l(p\epsilon)} \, .$$

In the following we concentrate on the s-wave solutions $l = 0$ near the critical coupling μ_c. Under the physically sensible condition $p\epsilon \ll 1$, for perturbative coupling at the cutoff $\mu^2 \ll 1$ and by using the expansion of the Bessel functions for small argument,

$$J_0(z) \simeq 1 - \frac{z^2}{4} \quad , \quad N_0(z) \simeq \frac{2}{\pi} \, \ln \frac{z}{2} \, ,$$

we find at the critical coupling $\mu = \mu_c$ the eigenvalue equation determining the massive spectrum

$$\frac{2}{\pi} \frac{J_0(pR)}{N_0(pR)} \, \ln \frac{pR}{2} \; = \; 1 \, , \tag{5.3}$$

up to negligibly small terms of order $p^2 \epsilon^2$. Interestingly enough, precisely at the critical coupling, the UV cutoff dependence of the KK masses cancels out from eq. (5.3). If we redo the same analysis slightly below the critical coupling, we find

$$\frac{2}{\pi} \frac{J_0(pR)}{N_0(pR)} \left(\ln \frac{pR}{2} + \frac{\mu_c^2 - \mu^2}{\mu_c^2} \ln \frac{R}{\epsilon} \right) = 1 \, , \tag{5.4}$$

and the KK masses start to depend on the cutoff.

In analogy to the lightest mass, the KK masses (5.4) can be expressed as functions of the running coupling $\mu^2(p)$ evaluated at the pole. Indeed, eq. (5.4) can be written as

$$\frac{J_0(pR)}{N_0(pR)} = \frac{1}{4} \, \mu^2(p) \, , \tag{5.5}$$

where the running coupling is given by eq. (2.3).

We note in passing that keeping subleading terms in the expansion of the Bessel functions, one obtains, near the critical coupling, the following equation instead of eq. (5.4),

$$\frac{2}{\pi} \frac{J_0(pR)}{N_0(pR)} \left(\ln \frac{pR}{2} + \frac{3}{16} + \frac{2\pi(\mu_c^2 - \mu^2)}{\mu_c^4} \right) = 1 \, .$$

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We then define the corrected value for the critical coupling $\hat{\mu}_c$ as

$$\hat{\mu}_c^2 = \mu_c^2 - \frac{3}{32\pi}\mu_c^4 \ ,$$

where μ_c by definition is given by eq. (2.4). Clearly, the second term here is subleading; it is suppressed, as compared to the first term, by $[\ln(R/\epsilon)]^{-1}$. After this qualification, eq. (5.4) remains valid, with $\hat{\mu}_c$ substituted for μ_c. Formulae in previous sections remain valid too, again with $\hat{\mu}_c$ used instead of μ_c everywhere. In the leading-log approximation one neglects the difference between $\hat{\mu}_c$ and μ_c. Thus, consistent leading-log approximation indeed corresponds to neglecting higher order terms in the expansion of J_0 and its derivatives.

It is likely that other physical quantities are also cutoff-independent at the critical coupling. For example, by using the results of Goldberger and Wise [2], we find the scalar propagator at the critical point. It was shown in ref. [2] that the (Euclidean) Dyson resummation of the scalar propagator, by including the brane localized mass insertions, in a mixed representation, 4d momentum space and 2d extra-dimensional coordinate space, is (cf. ref. [11])

$$\begin{aligned} G(p,y,y') &= D(p,y,y') + \mu^2 \ D(p,y,0)D(p,0,y') \\ &\quad + \mu^4 \ D(p,y,0) \ D(p,0,0)D(p,0,y') + \cdots \\ &= D(p,y,y') + \frac{\mu^2}{1-\mu^2 D(p,0,0)} \ D(p,y,0)D(p,0,y') \ , \end{aligned} \tag{5.6}$$

where $D(p,y,y')$ is the free scalar six-dimensional propagator. For the scalar field with the Dirichlet boundary condition at $r = R$, the propagator at the origin of the compact space $D(p,0,0)$ can be approximately evaluated, for different values of the four-dimensional momentum, as

$$\begin{aligned} D(p,0,0) &\sim \frac{1}{4\pi}\ln\frac{\Lambda^2}{p^2} \quad , &&\text{for } \ p^2 \gg 1/R^2 \ , \\ D(p,0,0) &\sim \frac{1}{4\pi}\ln(\Lambda^2 R^2) + O(p^2 R^2) \quad , &&\text{for } \ p^2 \ll 1/R^2 \ . \end{aligned} \tag{5.7}$$

Note that for $p^2 \gg 1/R^2$ the propagator (5.6) indeed exhibits logarithmic scale-dependence typical of renormalization group running of the coupling μ^2. It can then be easily shown that at the critical coupling $\mu = \mu_c$,

$$\begin{aligned} \frac{\mu_c^2}{1-\mu_c^2 D(p,0,0)} &= \frac{2\pi}{\ln(pR)} \ , &&\text{for } \ p^2 \gg \frac{1}{R^2} \ , \text{ and} \\ \frac{\mu_c^2}{1-\mu_c^2 D(p,0,0)} &\sim O\left(\frac{1}{p^2 R^2}\right) \ , &&\text{for } \ p^2 \ll \frac{1}{R^2} \ . \end{aligned} \tag{5.8}$$

These quantities are indeed independent of the UV cutoff. The existence of the massless mode is again obvious: it is seen as a pole in the propagator at $p^2 = 0$ for the critical coupling $\mu = \mu_c$.

6. Mixing with brane scalar: classical running and critical couplings

6.1 Running couplings

There are several reasons to try to generalize the previous 6d toy model by including brane-localized fields. In particular, we would like to better understand the possible role of brane fields in the value of the critical coupling(s) and also the dynamics of the field theory living on the brane close to the critical couplings. The next-to-simplest example includes, in addition to the bulk scalar ϕ, a brane-localized scalar H mixing with ϕ. The corresponding action is

$$S = \int d^4x d^2y \left(\frac{1}{2}(\partial_A \phi)^2 - \delta^2(y) \left[-\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - \frac{1}{2}(\partial_\mu H)^2 + \frac{h}{2}\phi H^2 + \frac{\lambda'}{4}H^4 \right] \right). \quad (6.1)$$

The brane scalar H is massless in the vacuum $\langle \phi \rangle = 0$. Notice that μ^2, h and λ' are dimensionless couplings and can naturally mix, whereas λ is dimensionfull and does not mix. It is straightforward to work out the classical (tree-level) field theory diagrams which contribute to the running of the coupling constants. We find the RG equations

$$Q\frac{d\mu^2}{dQ} = -\frac{1}{2\pi}\mu^4 \ ,$$

$$Q\frac{dh}{dQ} = -\frac{1}{2\pi}h\mu^2 \ ,$$

$$Q\frac{d\lambda'}{dQ} = +\frac{1}{4\pi}h^2 \ .$$

The integration of these RG equations gives

$$\mu^2(Q) = \frac{\mu^2}{1 + \frac{\mu^2}{2\pi}\ln\frac{Q}{\Lambda}} \ ,$$

$$h(Q) = \frac{h}{1 + \frac{\mu^2}{2\pi}\ln\frac{Q}{\Lambda}} \ ,$$

$$\lambda'(Q) = \lambda' + \frac{h^2}{4\pi}\frac{\ln\frac{Q}{\Lambda}}{1 + \frac{\mu^2}{2\pi}\ln\frac{Q}{\Lambda}} \ , \quad (6.2)$$

where $\mu \equiv \mu(\Lambda)$, etc. The running couplings $\mu^2(Q)$ and $h(Q)$ become strong at the critical coupling μ_c defined in the previous sections. Notice, however, that the coupling $\lambda'(Q)$ becomes negative and appears to destabilize the potential for smaller values of the bare coupling μ; actually, $\lambda' \to -\infty$ at the critical coupling! As we will see in a moment, of particular relevance for this problem is the combination of couplings $\bar{\mu}^2 \equiv \mu^2 + h^2/(2\lambda')$, which runs according to

$$\frac{1}{[\mu^2 + h^2/(2\lambda')](Q)} = \frac{1}{\mu^2 + h^2/(2\lambda')} + \frac{1}{2\pi}\ln\frac{Q}{\Lambda} \ . \quad (6.3)$$

Clearly we find here a *second critical coupling* defined by

$$\frac{1}{2\pi}[\mu^2 + h^2/(2\lambda')]_c \ \ln\frac{R}{\epsilon} = 1 \ . \quad (6.4)$$

Remarkably, it occurs at $\mu < \mu_c$ (we continue to use the notation μ_c for the first critical coupling). At the new critical coupling, $\mu^2(R^{-1})$ and $h(R^{-1})$ are finite, whereas

$$\lambda'(R^{-1}) = 0 .$$

The new critical coupling (6.4) is therefore the one above which the scalar potential appears destabilized.

We now turn to closer examination of the new critical coupling, expectation values of the fields, mass spectrum and the effective theory associated to it.

6.2 Simplified description near the new critical coupling

A simple but heuristic description near the new critical coupling is to regularize the delta function as above and write a 4d kinetic term for H inside the disk $r < \epsilon$. This would be consistent if the field equations implied that H is independent of r inside the resolved brane. We will see in the next section that with regularized δ-function, this actually is not the case. We will turn to the accurate treatment in the next section, and here we proceed with the somewhat heuristic analysis.

In this description, for $r < \epsilon$ the scalar potential is

$$V(r) = \frac{1}{\pi\epsilon^2}\left(-\frac{\mu^2}{2}\phi^2 + \frac{h}{2}\phi H^2 + \frac{\lambda'}{4}H^4 + \frac{\lambda}{4}\phi^4\right) , \tag{6.5}$$

whereas $V = 0$ for $r > \epsilon$. The classical field equations close to the critical coupling, by keeping only the relevant terms, are

$$\Delta^{(2)}\phi = 0 \quad , \qquad r > \epsilon ,$$

and

$$\Delta^{(2)}\phi + \frac{\mu^2}{\pi\epsilon^2}\phi - \frac{h}{2\pi\epsilon^2}H^2 = 0 ,$$
$$h\phi H + \lambda' H^3 = 0 \quad , \qquad r < \epsilon . \tag{6.6}$$

By combining the field equations, we find at $r < \epsilon$

$$\pi\epsilon^2 \Delta^{(2)}\phi + \left(\mu^2 + \frac{h^2}{2\lambda'}\right)\phi = 0 .$$

The analysis now is similar to that in section 3, but the role of the coupling is played by the combination $\bar{\mu}^2 \equiv \mu^2 + (h^2/2\lambda')$. This leads precisely to the condition (6.4) for the second critical coupling. In complete analogy to the analysis in section 3, we find that there is a light state whose mass close to the second critical coupling and $\bar{\mu} \le \mu_c$ is

$$p^2 = M^2 = \frac{8\pi}{R^2}\frac{\mu_c^2 - \bar{\mu}^2}{\mu_c^4} . \tag{6.7}$$

The scalar expectation values for $\bar{\mu} \ge \mu_c$ are given by

$$\phi_0^2 = \frac{\bar{\mu}^2 - \mu_c^2}{\lambda} \quad , \quad H_0^2 = -\frac{h}{\lambda'}\phi_0 . \tag{6.8}$$

Hence, the second critical coupling is the point of the second order phase transition, at which both ϕ and H obtain expectation values.

7. Resolving the delta-function in the model with brane scalar

7.1 Regularizing the delta-function

Let us now regularize the delta-function more carefully. We include a gradient term for H in the two extra dimensions to account for the radial variation of H inside the resolved brane. As before, the brane is a disk of radius ϵ. The action inside the brane ($r < \epsilon$) is

$$S_{in} = \int \ d^2y \ d^4x \left[\left(\frac{1}{2} (\partial_A \phi)^2 + \frac{\mu^2}{2\pi\epsilon^2} \phi^2 \right) \right.$$
$$\left. + \frac{1}{\pi\epsilon^2} \left(\frac{1}{2} (\partial_\mu H)^2 - \frac{1}{2} (\partial_a H)^2 + \frac{m^2}{2} H^2 - \frac{1}{2} h\phi H^2 - \frac{\hat{\lambda}'}{4} H^4 \right) \right] \ .$$

We are going to discuss the critical couplings, so the self-coupling of ϕ is unimportant (see below) and is neglected here. Since we interpret H as a brane field, its boundary condition at $r = \epsilon$ is Dirichlet. In order that the massless excitation of H above the background $H = 0$ be possible, we introduced a mass term $m^2 \sim \epsilon^{-2}$. This is part of our regularization procedure.

Note the "wrong", tachyonic sign of m^2 and also of μ^2. Note also that we introduced the coupling $\hat{\lambda}'$, which is not yet quite the quartic coupling constant in the delta-function limit.

Let us consider the linearized equation for H in the background $\phi = H = 0$,

$$\Delta^{(2)} H + m^2 H \equiv LH = -p^2 H \ . \tag{7.1}$$

We choose m^2 in such a way that this equation has a zero mode $p^2 = 0$ with the Dirichlet boundary condition at $r = \epsilon$ (we will not need the explicit expression for m^2 or the zero mode). Let

$$H^{(0)} = H^{(0)}(r/\epsilon)$$

denote the zero mode, normalized in such a way that

$$2 \int_0^1 \ \rho d\rho \ [H^{(0)}(\rho)]^2 = 1 \ , \tag{7.2}$$

where $\rho = r/\epsilon$. The low energy effective theory for the brane field H is obtained by writing

$$H(x,y) = H_B(x) \ H^{(0)}(y) \ .$$

Naively at least, this results in a brane field $H_B(x^\mu)$ with canonically normalized kinetic term, a δ-function interaction of H_B with ϕ with coupling constant h, and a quartic self-interaction of H_B with coupling constant

$$\lambda' = 2\hat{\lambda}' \int_0^1 \ d\rho\rho \ [H^{(0)}(\rho)]^4 \ . \tag{7.3}$$

This coupling differs from $\hat{\lambda}'$ by a numerical factor of order 1.

7.2 Phase transition at the second critical coupling

Let us now show that at the second critical coupling, the second order phase transition indeed occurs, so that above this point the fields ϕ and H develop expectation values. A signal for the phase transition is the existence of an approximate modulus near the origin in the field space, i.e., the direction along which the curvature of the effective scalar potential vanishes. In other words, at the phase transition point there exists a non-trivial x^μ-independent solution to the field equations with small (but otherwise arbitrary) amplitude. To see that this is indeed the case precisely at the second critical coupling, we neglect $\lambda \phi^4$ term in the action and write the field equations inside the resolved brane for the fields ϕ and H depending on r only,

$$\Delta^{(2)} H + m^2 H - h\phi H - \hat{\lambda}' H^3 = 0 \ , \tag{7.4}$$

$$\pi \epsilon^2 \Delta^{(2)} \phi + \mu^2 \phi - \frac{h}{2} H^2 = 0 \quad \text{for } r < \epsilon \ . \tag{7.5}$$

Outside the brane, the field ϕ still obeys the 2d Laplace equation.

Let us begin with eq. (7.4). This equation has the form

$$L \, H = J \ , \tag{7.6}$$

where

$$J = h\phi H + \hat{\lambda}' H^3$$

and the operator L is defined in (7.1). Since L has a zero mode $H^{(0)}(y)$, eq. (7.6) has a solution (with the Dirichlet condition at $r = \epsilon$) if and only if J is orthogonal to the zero mode,

$$\int_0^1 \rho d\rho \, H^{(0)} \, J = 0 \ . \tag{7.7}$$

If this property is satisfied, one can solve eq. (7.6) perturbatively, considering J as perturbation (given that $\Delta^{(2)}$ and m^2 are of order ϵ^{-2} inside the resolved brane). To the zeroth order one has

$$H(y) = H_0 \, H^{(0)}(y) \ , \tag{7.8}$$

where H_0 is a constant.

Let us now turn to eq. (7.5). Recalling that the boundary condition for ϕ at $r = \epsilon$ is

$$\frac{d\phi/d\rho}{\phi}(\rho = 1) = -\frac{1}{\ln \frac{R}{\epsilon}} \ , \tag{7.9}$$

one observes that ϕ is "large" (inside the brane) compared to $d\phi/d\rho$. Hence we write the solution by considering the last two terms in (7.5) as perturbations:

$$\phi(\rho) = \phi_0 + \frac{h}{2\pi} H_0^2 \int_0^\rho \frac{d\rho'}{\rho'} \int_0^{\rho'} \rho'' d\rho'' [H^{(0)}(\rho'')]^2 - \frac{\mu^2}{4\pi} \rho^2 \phi_0 + \cdots \ ,$$

where the leading order expression for $H(y)$, eq. (7.8) has been used. Since h and μ are small, the second and third terms here are indeed small compared to the first term on the right hand side.

In view of eq. (7.2), the boundary condition (7.9) gives

$$\frac{1}{2\pi}\left(\frac{h}{2}H_0^2 - \mu^2\phi_0\right) = -\frac{1}{\ln\frac{R}{\epsilon}}\phi_0 \ . \tag{7.10}$$

To find another relation between ϕ_0 and H_0, we use eq. (7.7). We insert the leading order expressions for ϕ (that is $\phi = \phi_0$) and H (that is $H = H_0 H^{(0)}(y)$) and obtain

$$h\phi_0 H_0 + \lambda' H_0^3 = 0 \ , \tag{7.11}$$

where λ' is given by eq. (7.3). Equations (7.10) and (7.11) are consistent precisely at the second critical coupling defined by eq. (6.4). Thus, we find that at the second critical coupling, there exists an approximate modulus near the origin of the field space. The second critical coupling is indeed the point of the second order phase transition.

The rest of the analysis parallels that of section 3. With $\lambda\phi^4$ term included, the expectation values of the fields just above the critical coupling are indeed given by eq. (6.8), while the new light state has mass (6.7). We conclude that our more accurate analysis confirms the results of the heuristic treatment of subsection 6.2.

8. Conclusions

The toy models we analyzed in this paper possess fairly rich physics: running couplings, phase transitions, spontaneous symmetry breaking and infrared strong dynamics which all occur at the level of classical field theory. In addition of being an interesting laboratory for studying difficult physics issues such as the ones just mentioned, our models could be of some relevance for phenomenological string or extra dimensional models with small radii and large (close to the Planck mass) fundamental mass scale. Indeed, if for some (eventually dynamical) reason the microscopic parameters are very close to the point (surface) defining the critical coupling(s), very light modes and standard four dimensional physics are generated out of a higher dimensional theory with large scales only. This mechanism could be of relevance for the problem of electroweak symmetry breaking and mass generation or/and for explaining the smallness of supersymmetry breaking in appropriate supersymmetric extensions. Of course, an important step to make before addressing these phenomenological issues is the inclusion of other fields like chiral 4d fermions and gravitational interaction.

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